PH 561.4

Reg No :

St Aloysius College (Autonomous)
Mangaluru
SEMESTER IV - PG EXAMINATION - M.Sc Mathematics
April - 2025
Measure Theory and Integration

Time : 3 Hours   Max. Marks : 70

Answer FIVE FULL questions   (14x5=70)

1. a. If $\{f_n\}$ is a sequence of measuable functions defined on the same measurable set then show that $\sup f_n$ and $\inf f_n$ are measurable. 6

   b. If f, g are real valued measurable functions defined on the same measurable set E and c is any real number, then prove that $f+c$, fc and $f+g$ are measurable. 8

2. a. Show that if E is measurable, then E^c is measurable. 2

   b. Show that if $m^*(E) = 0$ then E is measurable. 4

   c. Let $\{E_i\}$ be a sequence of measurable sets. If $E_1 \subseteq E_2 \subseteq E_3 \subseteq \ldots$ then prove that $m(\lim E_i) = \lim m(E_i)$. 8

3. a. State and prove the countable subadditive property of the Lebesgue outer measure. 6

   b. Define essential supremum of a measurable function. Prove that $f \leq \operatorname{ess\,sup} f\ a.e.$ for any measurable function. 8

4. a. State Fatou's lemma. State and prove the Lebesgue monotone convergence theorem. 6

   b. If f and g are integrable functions then prove the following: 8

      1. af is integrable and $\int af\, dx = a\int f\, dx$, where $a \in \mathbb{R}$.
      2. $f+g$ is integrable and $\int (f+g)\, dx = \int f\, dx + \int g\, dx$.

5. a. If ϕ is a measurable simple function taking the distinct values $a_1, a_2, \ldots, a_n$ on the set $A_i = \{x : \phi(x) = a_i\}$ then prove the following: 6

      1. $\int_E \phi\, dx = \sum_{i=1}^{n} a_i\, m(A_i \cap E)$ for any measurable set E.
      2. $\int_{AUB} \phi\, dx = \int_A \phi\, dx + \int_B \phi\, dx$ for any disjoint measurable sets A and B.

   b. Show that if f and g are measurable, $|f| \leq |g|\ a.e.$ and g is integrable then f is integrable. 8

6. a. State and prove Minkowski's inequality. 7

   b. State and prove Holder's inequality. 7

7. a. Show that $L^p(X,\mu)$ is a vector space over $\mathbb{R}$. Show that if $\mu(X) < \infty$ and $0 < p < q \leq \infty$ then $L^q(\mu) \subseteq L^p(\mu)$. 8

   b. Prove that every convex function on an open interval is continuous. 6

8. State and prove the Hahn decomposition theorem. 14

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PH 562.4 Reg No : [ ]

St Aloysius College (Autonomous)

Mangaluru

SEMESTER IV - PG EXAMINATION - M.Sc Mathematics

April - 2025

Complex Analysis II

Max. Marks : 70

Time : 3 Hours

(14x5=70)

Answer FIVE FULL questions

1. Prove that a region is simply connected if and only if $n(\gamma, a) = 0$ for all cycles γ in Ω and all points a which do not belong to Ω. 14

2. a. Let $f(z)$ be analytic except for isolated singularity a_j in a region Ω then prove that $\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_j n(\gamma, a_j)Res_{z=a_j}f(z)$ for any cycle γ which is homologous to zero in Ω and does not pass through any of the points a_j. 8

   b. State and prove the Argument principle. 6

3. a. Evaluate $\int_c \frac{cosz}{(z-\pi i)^2}dz, c: |z| = 5$. 4

   b. Evaluate $\int_{-\infty}^{\infty} \frac{cosx}{x^2+a^2}dx, a$ is real. 5

   c. Evaluate the integral $\int_c \frac{z^2+4}{z^3+2z^2+2z}dz, (i)\ c: |z| = 1, (ii)\ c: |z-1| = 5$. 5

4. State and prove the Reflection Principle. 14

5. a. If u_1 and u_2 are harmonic in a region Ω, then prove that $\int_\gamma u_1 * du_2 - u_2 * du_1 = 0$ for every cycle γ which is homologous to zero in Ω. 6

   b. Prove that a non-constant harmonic function has neither a maximum nor a minimum in its region of definition. Consequently prove that the maximum and minimum on a closed bounded set E are taken on the boundary of E. 8

6. a. State and prove Weirstrass theorem. 8

   b. If $f(z)$ is analytic in a region Ω containing z_0, then show that the representation $f(z) = f(z_0) + f'(z_0)\frac{z-z_0}{1!} + \ldots + f^n(z_0)\frac{(z-z_0)^n}{n!} + \ldots$ is valid in the largest open disk of center z_0 contained in Ω. 6

7. a. Show that $lim_{n\to\infty}(1+\frac{z}{n})^n = e^z$ uniformly on every compact subset of complex plane. 6

   b. If $f(z)$ is analytic in an annulus $R_1 < |z-a| < R_2$ then prove that $f(z)$ can be developed in a general power series of the form $f(z) = \sum_{n=-\infty}^{\infty} A_n(z-a)^n$ where $A_n = \frac{1}{2\pi i}\int_{|\tau-a|=r}\frac{f(\tau)}{(\tau-a)^{n+1}}d\tau$. 8

8. a. Show that $\frac{\pi^2}{sin^2\pi z} = \sum_{n=-\infty}^{\infty}\frac{1}{(z-n)^2}$. Deduce that $\pi cotz = \frac{1}{z} + \sum_{n\neq 0}[\frac{1}{z-n} + \frac{1}{n}]$. 8

   b. i) Show that a necessary and sufficient condition for the absolute convergence of the infinite product $\prod_{n=1}^{\infty}(1+a_n)$ is the convergence of the series $\sum_{n=1}^{\infty}|a_n|$. 6
   ii) Prove that $f(z)$ is an entire function with a finite number of zeroes $a_1, a_2, \ldots a_n$ and a zero of order m at the origin if and only if $f(z)$ is of the form $f(z) = z^m e^{g(z)}\prod_{j=1}^{n}(1-\frac{z}{a_j}), 0 \neq a_j \in \mathbb{C}, m \geq 0$ and $g(z)$ is an entire function.

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PS 564.4

Reg No : [ ]

St Aloysius College (Autonomous), Mangaluru
SEMESTER IV - PG EXAMINATION - M.Sc Mathematics
April - 2025
Functional Analysis

Time : 3 Hours | Max. Marks : 70

Answer FIVE FULL questions (14x5=70)

1. a. Let X be a complete metric space and $\{F_n\}$ be a decreasing sequence of nonempty closed sets in X such that $d(F_n) \to 0$. Then prove that $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point. 6

   b. State and prove the Baire's category theorem. 8

2. a. Let N and N' be normed linear spaces. Prove that the following are equivalent for a linear transformation $T : N \to N'$ 8
   1. T is continuous
   2. T is continuous at the origin
   3. There exists a real number $K \geq 0$, such that $||Tx|| \leq K||x||$ for all $x \in N$.
   4. If $S = \{x \in N : ||x|| \leq 1\}$ then $T(S)$ is a bounded set in N'.

   b. Let L be a linear space made into a normed linear space by $||.||$ and $||.||'$. Show that these two norms are equivalent if and only if there exist positive reals K_1 and K_2 such that $K_1||x|| \leq ||x||' \leq K_2||x||$, for all $x \in L$. 6

3. a. Prove that a nonzero normed linear space N is a Banach space if and only if $\{x \in N : \|x\| = 1\}$ is complete. 6

   b. Prove that a linear transformation of l_p^n , where $p \geq 1$, into an arbitrary normed linear space is continuous. Let N be a finite dimensional normed linear space with dimension $n > 0$ and let $\{e_1, e_2, \ldots, e_n\}$ be a basis of N. Then show that the map $T : N \to l_1^n$ given by $T(x) = (x_1, x_2, \ldots, x_n)$ whenever $x = x_1e_1 + x_2e_2 + \ldots + x_ne_n$ uniquely, is continuous. 8

4. a. Let B be a Banach space and N be a normed linear space. If $\{T_i\}$ is a nonempty set of continuous linear transformations of B into N with the property that $\{T_i(x)\}$ is a bounded subset of N for each vector $x \in B$, then prove that $\{||T_i||\}$ is a bounded set of numbers. 6

   b. Define the conjugate space N^* of a normed linear space N. Show that N can be embedded in N^{**}. 8

5. a. If M and N are closed linear subspaces of a Banach space B such that $B = M \oplus N$ then prove that there exists a projection P on B whose range is M and nullspace is N. 10

   b. If B and B' are Banach spaces and if T is a continuous linear transformation of B into B' then prove that the graph of T is closed in $B \times B'$. 4

6. Let $\{e_i\}_{i \in I}$ be an orthonormal set in a Hilbert space H. For any vector $x \in H$, prove that $\sum_{i=1}^{n} |\langle x, e_i \rangle|^2 \leq \|x\|^2$. 14

7. a. Define a unitary operator on a Hilbert space H. Is an isometric isomorphism of H into itself a unitary operator on H? Justify. 5

   b. Let H be a Hilbert space. Prove that an operator T on a Hilbert space is self-adjoint if and only if $\langle Tx, x \rangle$ is real for all $x \in H$. 4

   c. State and prove the parallelogram law in a Hilbert space H. Does the Parallelogram law hold in the Banach space l_1^n? Justify your answer. 5

8. a. If M is a proper closed linear subspace of a Hilbert space H, then prove that there exists a nonzero vector z_0 in H such that $z_0 \perp M$. 6

   b. Define the orthogonal complement $M^{\perp}$ of a subspace M of a Hilbert space H. Show that $M^{\perp}$ is a closed linear subspace of H. If M is a closed linear subspace of a Hilbert space H, then prove that $H = M \bigoplus M^{\perp}$. 8

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PS 565.4 Reg No :

St Aloysius College (Autonomous)

Mangaluru

SEMESTER IV - PG EXAMINATION - M.Sc Mathematics

April - 2025

PARTIAL DIFFERENTIAL EQUATIONS

Time : 3 Hours Max. Marks : 70

Answer FIVE FULL questions (14x5=70)

1. a. Prove necessary and sufficient condition that there exists between two functions $U(x,y)$ and $V(x,y)$ a relation $F(U,V)=0$ not involving x or y explicitly is $\frac{\partial(U,V)}{\partial(x,y)}=0$. 6

   b. Find the integral curves of $\frac{dx}{y^2}=\frac{dy}{-xy}=\frac{dz}{x(z-2y)}$. 8

2. a. Prove that a necessary and sufficient condition for a differential equation $P(x,y,z)dx+Q(x,y,z)dy+R(x,y,z)dz=0$ to be integrable is that $X.CurlX=0$. 8

   b. Test for integrability of $yz\,dx+2xz\,dy-3xy\,dz=0$ and find its primitive. 6

3. a. Find the characteristic of the equation $pq=z$ and determine the integral surface which passes through the parabola $x=0$ and $y^2=z$. 8

   b. Obtain the partial differential equation by eliminating the arbitrary function f from the following, 6

      i) $f(x^2+y^2+z^2, z^2-2xy)=0$.

      ii) $u=f(\frac{xy}{z})$.

4. a. Find the general solution of $x(y^2-z^2)p-y(z^2+x^2)q=(x^2+y^2)z$. 6

   b. Find the integral surface of the linear partial differential equation $2y(u-3)u_x+(2x-u)u_y=y(2x-3)$ which passes through the circle $x^2+y^2=2x, u=0$. 8

5. a. Solve $(D^2-4DD'+4D'^2)u=e^{2x+y}$. 6

   b. Find the family of surfaces which is orthogonal to one-parameter family of surfaces $z(x+y)=c(3z+1)$, where c is a constant and which passes through the circle $x^2+y^2=1, z=1$. 8

6. a. Solve $(D^2+DD'-6D'^2)u=ycosx$. 8

   b. Find the family of surfaces which is orthogonal to one-parameter family of surfaces $x(x^2+y^2+z^2)=cy^2$, where c is a constant. 6

7. a. Classify the equation $u_{xx}-2sinxu_{xy}-cos^2xu_{yy}-cosxu_y=0$ and reduce it into cannonical form. 8

   b. Obtain the D'Alemberts solution of the initial value problem of Cauchy type described as $z_{tt}-C^2z_{xx}=0, \quad -\infty<x<\infty, t>0$, initial conditions 6

      $z(x,0)=f(x), z_t(x,0)=g(x)$, where f and g are twice continuously differentiable functions on $\mathbb{R}$.

8. a. A stretched string of finite length L is held fixed at its ends and is subjected to an initial displacement $u(x,0)=u_0sin(\frac{\pi x}{L})$. The string is released from this position with zero initial velocity. Find the resultant time dependent motion of the string. 6

   b. Classify the equation $y^2u_{xx}-x^2u_{yy}=0$ and reduce it to canonical form . 8

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