PH 561.3 Reg No : ☐☐☐☐☐☐☐☐

St Aloysius College (Autonomous)

Mangaluru

M.Sc. Mathematics - SEMESTER III PG EXAMINATION

NOVEMBER - 2024

Complex Analysis I

Time : 3 Hours **Max. Marks : 70**

Answer FIVE FULL questions **(14x5=70)**

1. a. What is the complex number which corresponds to the point $\left(\frac{1}{2}, \frac{1}{2}\frac{1}{\sqrt{2}}\right)$ of the Riemann sphere under the stereographic projection? 4

   b. Find the point on the Riemann sphere which corresponds to the complex number $2+i$ under the stereographic projection. 5

   c. Show that $|a+b|^2 = |a|^2 + |b|^2 + 2\,\mathrm{Re}\, a\bar{b}$ where $a, b \in \mathbb{C}$. Also show that if $a, b \in \mathbb{C}$ then $|a+b| = |a| + |b|$ if and only if $a\bar{b} \geq 0$. 5

2. a. Prove that an analytic function in a region Ω whose modulus is a constant must reduce to a constant. 4

   b. Prove that the real and imaginary part of analytic function are harmonic with an example. 5

   c. Expand $f(z) = \frac{1}{z}$ in powers of z in $|z| < 1$. 5

3. a. Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ 4

   b. If $\sum_{n=0}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$ is a power series, then show that there exists R with $0 \leq R \leq \infty$ such that the series converges absolutely for every z with $|z| < R$, the sum of the series is an analytic function in $|z| < R$. 10

4. a. Show that the derivative of an analytic function in a region Ω is analytic in Ω. 10

   b. Let $f(z)$ be analytic in the region Δ' obtained from open disk Δ by omitting a finite number of interior points $\zeta_1, \zeta_2, \cdots, \zeta_n$. Let $\lim_{z \to \zeta_i} (z - \zeta_i) f(z) = 0$, for each $i, 1 \leq i \leq n$, then prove that $\int_\gamma f(z)dz = 0$, for any closed curve γ in Δ'. 4

5. a. Define index of a point with respect to an arc γ. Show that the index of a with respect to a piecewise differentiable closed curve γ which does not pass through a is constant in each of the regions determined by γ and $n(\gamma, a) = 0$ for any point a in the unbounded region. 8

   b. Use Cauchy's integral formula to evaluate $\int_C \frac{1}{z(z-1)} dz$ where C is the circle $|z| = 3$. 6

6. a. Let T be a linear transformation which fixes $1, 0$ and ∞, then prove that T must be identity. Let T be any linear transformation which maps z_2, z_3 and z_4 to $1, 0$ and ∞ respectively, then prove that T is unique. Also prove that cross ratio is invarient under linear transformation. 8

   b. Show that the cross ratio (z_1, z_2, z_3, z_4) is real if and only if the four points lie on a circle or on a straight line. 6

7. a. If $p(x, y)$ and $q(x, y)$ are real or complex valued continuous functions defined in a region Ω and if γ is any curve in Ω, then show that $\int_\gamma p dx + q dy$ depends only on the end points of γ if and only if there exists a function $u(x, y)$ in Ω with $\frac{\partial u}{\partial x} = p$ and $\frac{\partial u}{\partial y} = q$. 8

   b. If the piecewise differentiable closed curve γ does not pass through the point a then show that the value of the integral $\int_\gamma \frac{dz}{z-a}$ is a multiple of $2\pi i$. 6

8. a. If $f(z)$ is continuous on a closed and bounded set E and analytic in the interior of E, then show that the maximum of $|f(z)|$ on E is assumed on the boundary of E. 8

   b. Show that a non-constant analytic function maps open sets onto open sets 6

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St Aloysius College (Autonomous)
Mangaluru
Semester III – P.G. Examination – M. Sc. Mathematics
NOVEMBER - 2024
Topology

Time : 3 Hours **Max. Marks : 70**

Answer FIVE FULL questions **(14x5=70)**

1. a. Define a closed set in a topological space X. Let Y be a subspace of X. Prove that a subset A of Y is closed in Y if and only if $A = C \cap Y$ for some closed set C in X. 6

   b. Define a derived set. Show that a subset of a topological space is closed if and only if it containts all its limit points. 8

2. a. Define a basis for a topology. Let $\mathscr{C}$ be a collection of open subsets of a topological space X such that for each $x \in X$ and each open set U of X containing x there is an element C of $\mathscr{C}$ such that x∈C⊆U, then show that $\mathscr{C}$ is a basis for the topology of X. 8

   b. Define a Hausdorff space. If X is a Hausdorff space, then prove that every finite subset of X is closed in X . 6

3. a. Let Y be a subspace of X . Let $A \subseteq Y$ and let $\bar{A}$ denote the closure of A in X . Then prove that the closure of A in Y is $\bar{A} \cap Y$. 6

   b. Let X be a metric space with metric d. Define $\bar{d} : X \times X \to \mathbb{R}$ by $\bar{d}(x, y) = min\{d(x, y), 1\}$. Then show that $\bar{d}$ is a metric. 8

4. a. Prove the following : 4
   1. If $f : X \to Y$ and $g : Y \to Z$ are continuous then the map $g \circ f : X \to Z$ is continuous.
   2. If $f : X \to Y$ is continuous and if A is a subspace of X, then the restricted function $f_{|_A} : A \to Y$ is continuous.

   b. Let $f : X \to Y$ be continuous. If Z is a subspace of Y containing $f(X)$, then prove that the function $g : X \to Z$ obtained by restricting the range of f is continuous. 5

   c. Prove that the map $f : X \to Y$ is continuous if X can be written as the union of open sets U_α such that $f_{|_{U_\alpha}}$ is continuous for each α . 5

5. a. Prove that if the sets C and D form a separation of X and if Y is a connected subspace of X , then Y lies entirely within either C or D . 5

   b. Let $f : X \to Y$ be a continuous map of the compact metric space (X, d_X) to the metric space (Y, d_Y) . Then show that f is uniformly continuous. 5

   c. Define path connectedness. Prove that every path connected space is a connected space. 4

6. a. Define a locally compact space. If X is a Hausdorff space then prove that X is locally compact space if and only if, for all $x \in X$, there exists a neighbourhood U of x such that $\bar{U}$ is compact. 6

   b. State and prove intermediate value theorem. 8

7. a. Show that every compact hausdorff space is normal. 5

   b. State Urysohn lemma and Tietze extension theorem. 4

   c. Show that if X is second countable, then X is Lindelof. 5

8. a. Prove that every regular space with a countable basis is normal. 6

   b. Prove the following : 8
   1. Every regular space is Hausdorff.
   2. Every normal space is regular.

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St Aloysius College (Autonomous)
Mangaluru
M.Sc. Mathematics - SEMESTER III PG EXAMINATION
NOVEMBER - 2024
Numerical Analysis with Computational Lab

Time : 3 Hours Max. Marks : 70

Answer FIVE FULL questions (14x5=70)

1. a. Derive the Bisection method formula. 6

   b. Derive the Newton-Raphson method to find the root of an equation $f(x) = 0$. Further using the same method find the root of $f(x) = x + logx - 2 = 0$ with $x_0 = 1.5$. 8

2. a. Perform 3 iterations of the Bairstow method to extract the quadratic factor $x^2 + pq + q$ from the polynomial $P(x) = x^3 + x^2 - x + 2$. Use initial approximation $p_0 = -0.9$ and $q_0 = 0.9$ 6

   b. Find the smallest eigen value in magnitude of the matrix 8
   $$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$ using inverse power method.

3. a. Estimate the eigen values of the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ using the Gerschgorin bounds. 6

   b. Using Chebyshev method find the smallest positive root of the equation $f(x) = x^4 - 3x^3 - 10$ with $x_0 = 0.5$. Find the order of Newton-Raphson method in the iteration method $x_{k+1} = \phi(x_k)$ where $k = 0, 1, 2, \ldots$. 8

4. a. Given the following values of $f(x)$ and $f'(x)$. Estimate the value of $f(-0.5)$ using Hermite interpolation. 8

x	$f(x)$	$f'(x)$
-1	1	-5
0	1	1
1	3	7

   b. For the following data calculate the forward and backward difference polynomials and evaluate at $x = 0.5$ and $x = 0.35$. 6

x	0.1	0.2	0.3	0.4	0.5
$f(x)$	1.40	1.56	1.76	2.0	2.28

5. a. Derive Lobatto 3-point formula. 8

   b. Deduce the one-point and two-point formula for Gauss-Hermite integration method. Also evaluate $\int_{-\infty}^{\infty} \frac{e^{-x^2}}{x^2+x+1}$ using Gauss Hermite 2-point formula. 6

6. a. Evaluate the integral $I = \int_0^1 \frac{dx}{2x^2+2x+1}$ by using Lobatto 3-point formula and Radau 3-point formula. Compare with the exact solution. 8

   b. Evaluate the integral $I = \int_0^1 \frac{dx}{1+x}$ using composite Trapezoidal rule with 4 sub intervals. 6

7. a. Solve the initial value problem $y' = x + y, y(0) = 1$. Estimate $y(0.4)$ using Huen's method with $h = 0.1$. 6

   b. Solve the initial value problem $u' = -2tu^2$, $u(0) = 1, h = 0.2$ in the interval $[0, 0.4]$ using backward Euler method. 8

8. a. Solve the initial value problem $y' = x - y, y(0) = 1, h = 0.1$ using fourth order R-K method to find $y(0.3)$. 8

   b. Solve the initial value problem $y' = xy, y(0) = 1$. Using Euler's method with $h = 0.2$ in $[0, 1]$. 6

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St Aloysius College (Autonomous)
Mangaluru
Semester III – P.G. Examination – M.Sc. Mathematics
November - 2024
Commutative Algebra

Time: 3 hrs. Max Marks: 70

Answer FIVE FULL questions (14x5=70)

1. a. If $I_1, I_2, \ldots, I_n$ are ideals in a ring A with $I_j + I_k = A$ for all $1 \leq j \neq k \leq n$, then show that $\prod_{j=1}^{n} I_j = \bigcup_{j=1}^{n} I_j$. 6

   b. In a ring A, let $I_1, \ldots, I_n$ be ideals and let P be a prime ideal containing $\bigcap_{j=1}^{n} I_j$, then prove that $P \supseteq I_j$ for some $j, 1 \leq j \leq n$. 3

   c. Define an idempotent element in a ring A. Show that only idempotents in a local ring are 0 and 1. 5

2. a. Prove that the nilradical of a ring A is the intersection of all prime ideals of A. 10

   b. Prove that in a ring every nilpotent element is a zero-divisor. Is the converse true? justify your answer. 4

3. a. Define the Jacobson radical $J(A)$ of a ring A. If U denotes the class of all units in A, then show that $J(A) = \{x \in A \,:\, 1 - xy \in U \;\forall\, y \in A\}$. 5

   b. For the ideals I and J in a ring A, prove that $r(I \cap J) = r(I) \cap r(J)$. 4

   c. If D denotes the set of all zero divisors of a ring A, then prove that $D = \bigcup_{x \neq 0} r\left(\mathrm{Ann}(x)\right)$. 5

4. a. Let M be a finitely generated $A-$ module and I be an ideal of$A$. Let ϕ be an $A-$module endomorphism of M such that $\phi(M) \subseteq IM$. The prove that ϕ satisfies an equation of the form $\phi^n + a_1\phi^{n-1} + \ldots + a_{n-1}\phi + a_n$, where $a_i \in I$ for each $i, 1 \leq i \leq n$. 8

   b. Prove that a nonzero $A-$ module M is isomorphic to a quotient of A^n for some $n \in \mathbb{N}$ if and only if M is a finitely generated $A-$module. 6

5. a. Describe the localization of a ring A at a prime ideal P of A. 5

   b. Let A be a ring and let M be an $A-$module. For a prime ideal P, let $\mathcal{M}_P$ denote the $S^{-1}A-$module $S^{-1}M$, where $S = A - P$. Prove that the following statements are equivalent: 5
      1. $M = 0$
      2. $\mathcal{M}_P = 0$ for each prime ideal P of A
      3. $\mathcal{M}_m = 0$ for each maximal ideal m of A

   c. If N, P are submodules of an $A-$module M, and S is a multiplicatively closed set in A, then prove that $S^{-1}(N + P) = (S^{-1}N + S^{-1}P)$. 4

6. a. Let $g : A \to B$ be a ring homomorphism and S be a multiplicatively closed set in A such that $g(s)$ is a unit in B for every $s \in S$. Prove that there is a unique ring homomorphism $h : S^{-1}A \to B$ such that $g = h \circ f$, where $f : A \to S^{-1}A$ given by $f(a) = \frac{a}{1}, a \in A$. 6

   b. Let $f : A \to B$ be a ring homomorphism and let P be a prime ideal of A. Prove that P is a contraction of a prime ideal of B if and only if $P^{ec} = P$. 8

7. a. Let M, M' and M'' be $A-$modules, and S be multiplicatively closed in A. If $M' \xrightarrow{f} M \xrightarrow{g} M''$ is exact at M, then show that $S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M''$ is exact at $S^{-1}M$. 4

   b. Let I be an ideal of a ring A, and let $S = 1 + I$. Show that S is a multiplicatively closed subset of A. Further, show that $S = 1 + I$ is contained in the Jacobson radical of $S^{-1}A$. 5

   c. If I, J are ideals of a ring A, with J finitely generated then show that $S^{-1}(I : J) = (S^{-1}I : S^{-1}J)$. 5

8. a. Define a $P-$ primary ideal in a ring. Show that the intersection of finitely many $P-$primary ideals is $P-$primary. 5

   b. Define an irreducible ideal in a ring. In a Noetherian ring A prove that every ideal is a finite intersection of irreducible ideals. 5

   c. Let Q be a $P-$primary ideal in a ring A, x an element of A. Then prove the following: 4
      i) if $x \in Q$ then $(Q : x) = (1)$
      ii) if $x \notin Q$ then $(Q : x)$ is $P-$primary, and therefore $r(Q : x) = P$

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